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BRAUER–MANIN OBSTRUCTION TO WEAK APPROXIMATION ON ABELIAN VARIETIES

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ABSTRACT

Let A be an abelian variety defined over a number field K. Assume that the Tate–Shafarevich group is finite. We prove that the condition that the topological closure of $A(K)$ in $\prod_{v\in M_K^\infty} A(K_v)$ is open is equivalent to the condition that the Brauer–Manin obstruction is the only obstruction to weak approximation.

1. Introduction

Colliot-Thélène and Sansuc [4] gave the first example of the failure of weak approximation for a Del Pezzo surface of degree 4 and pointed out that the Brauer–Manin obstruction [7] is responsible for most known conterexamples to weak approximation. Let V be a smooth algebraic variety defined over a number field K with $V(K) \neq \emptyset$. If the Brauer–Manin obstruction to weak approximation is the only obstruction for V (Definition 2.1), then the topological closure of $V(K)$ in $\prod_{v\in M_K^\infty} V(K_v)$ is open, in particular, the K-rational points are Zariski dense in V (Lemma 2.3).

In this paper, we show that the converse is true for abelian varieties under the standard assumption that the Tate–Shafarevich group is finite.

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THEOREM 1.1: *Let A be an abelian variety defined over K. Assume that the topological closure of $A(K)$ in $\prod_{v\in M_K^\infty} A(K_v)$ is open. If the Tate–Shafarevich group is finite, then the Brauer–Manin obstruction to weak approximation is the only obstruction.*

When a variety is defined over $\mathbf{Q}$, Mazur made the following conjecture [9].

CONJECTURE 1.2 (Mazur): *Let V be a smooth algebraic variety defined over $\mathbf{Q}$ whose rational points $V(\mathbf{Q})$ are Zariski dense in V. Then the topological closure of $V(\mathbf{Q})$ in $V(\mathbf{R})$ is open.*

This conjecture together with Theorem 1.1 implies the following result.

PROPOSITION 1.3: *Let A be an abelian variety defined over $\mathbf{Q}$ whose rational points are Zariski dense. Assume that Conjecture 1.2 is true for A and that the Tate–Shafarevich group is finite. Then the Brauer–Manin obstruction to weak approximation is the only obstruction for A.*

COROLLARY 1.4: *Let A be a simple abelian variety of dimension d which is defined over $\mathbf{Q}$. Suppose the Mordell–Weil rank of A is at least $d^2 - d + 1$. If the Tate–Shafarevich group of A is finite, then the Brauer–Manin obstruction is the only obstruction to Weak Approximation.*

Proof: Waldschmidt [18] proved Conjecture 1.2 for such abelian varieties. ∎

COROLLARY 1.5: *Let E be a modular elliptic curve over $\mathbf{Q}$. Let $L(s)$ be the Hasse–Weil L-function for E over $\mathbf{Q}$. If $\mathrm{ord}_{s=1} L(s) = 1$, then the Brauer–Manin obstruction to weak approximation on E is the only obstruction.*

Proof: Kolyvagin [6] proved that the Tate-Shafarevich group of such an elliptic curve over $\mathbf{Q}$ is finite, and $\mathrm{rank} E(\mathbf{Q}) = \mathrm{ord}_{s=1} L(s) = 1$. Hence $E(\mathbf{Q})$ is Zariski dense in E. Since Mazur's Conjecture 1.2 is true for curves [9, §2], we can apply Proposition 1.3. ∎

Notice that an abelian variety does not satisfy weak approximation, not even weak weak approximation [14, p. 30, p. 20]. The proof of Theorem 1.1 uses the Tate global and local duality, Serre's result on congruence subgroups and Goursat's Lemma.

The analogue of Conjecture 1.2 in a higher number field does not hold. We will construct elliptic curves E over quadratic fields K with positive Mordell–Weil rank such that the topological closure of $E(K)$ in $\prod_{v\in M_K^\infty} E(K_v)$ is not open.

Hence the Brauer–Manin obstruction to weak approximation is not the only one for such E. This gives rise to an interesting problem: find new obstructions to weak approximation on abelian varieties defined over a number field.

The weak approximation is usually studied together with the Hasse Principle. For complete group varieties, the following result is known.

THEOREM 1.6: *Let V be a smooth variety defined over a number field K such that $V \otimes_K F$ is an abelian variety for some finite extension F of K. Assume that $Ш(Alb(V))$ is finite, then the Brauer–Manin obstruction for V to the Hasse Principle is the only one (Definition 2.1). In other words, if $V(K_v) \neq \emptyset$ for all places v of K and there is no Brauer–Manin obstruction, then there exists a global point $P \in V(K)$.*

Proof: When $\dim(V) = 1$, the statement is a consequence of [8, CH VI, §41, Thm 41.24]. The proof for the general case is similar. ∎

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2. Preliminaries

2.1 THE BRAUER–MANIN OBSTRUCTION. Let M_K be a complete set of absolute values of a number field K and M_K^f (resp. M_K^∞) the set of non-archimedean (resp. archimedean) absolute values. For any $v \in M_K$, let K_v be the completion of K with respect to v. Denote by $\mathbf{A}_K$ the adèle ring of K. Suppose that V is a smooth projective variety defined over K and $\mathbf{Br}(V)$ is the Brauer group of V/K. By local class field theory, there is a natural continuous right-linear pairing [3]

$$(1) \qquad \begin{aligned} V(\mathbf{A}_K) \times \mathbf{Br}(V) &\longrightarrow \mathbf{Q}/\mathbf{Z}, \\ ((x_v), b) &\longrightarrow \sum_v \mathrm{inv}_v b^{(v)}(x_v). \end{aligned}$$

By global class field theory, the restriction of the pairing (2.1) to $V(K) \times \mathbf{Br}(V)$ is trivial. We denote by $V(\mathbf{A}_K)^{\mathbf{Br}}$ the left kernel of the above pairing, that is,

$$V(\mathbf{A}_K)^{\mathbf{Br}} := \{(x_v) \in V(\mathbf{A}_K) \mid ((x_v), b) = 0, \text{ for any } b \in \mathbf{Br}(V)\}.$$

Then $V(K)$ is embedded in $V(\mathbf{A}_K)^{\mathbf{Br}}$ and the closure $\overline{V(K)}$ of $V(K)$ in $V(\mathbf{A}_K)$ is contained in $V(\mathbf{A}_K)^{\mathbf{Br}}$.

Definition 2.1: Assume that $V(K_v) \neq \emptyset$ for all v. We say that there is **no Brauer–Manin obstruction to the Hasse Principle** if $V(\mathbf{A}_K)^{\mathbf{Br}} \neq \emptyset$. We say that **the Brauer–Manin obstruction to the Hasse Principle is the only obstruction** if the condition $V(\mathbf{A}_K)^{\mathbf{Br}} \neq \emptyset$ implies that $V(K) \neq \emptyset$. When $V(K) \neq \emptyset$, we say that **the Brauer–Manin obstruction is the only obstruction to weak approximation** for V if $\overline{V(K)} = V(\mathbf{A}_K)^{\mathbf{Br}}$.

LEMMA 2.2: *Let W, Y_1 and Y_2 be metrizable topological spaces. Assume that there are two embeddings $i_1 : W \to Y_1$, $i_2 : W \to Y_2$. If Y_2 is compact, then the closure of $i_1 \times i_2(W)$ in $Y_1 \times Y_2$ is projected onto the closure of $i_1(W)$ in Y_1.*

Proof: Denote by Pr_1 the projection from $Y_1 \times Y_2$ to Y_1. Since Pr_1 is continuous, we obtain $\mathrm{Pr}_1(\overline{i_1 \times i_2(W)}) \subseteq \overline{\mathrm{Pr}_1(i_1 \times i_2(W))} = \overline{i_1(W)}$.

For any $v \in \overline{i_1(W)}$, choose a sequence $\{w_n\} \subset W$ such that $i_1(w_n) \to v$ as $n \to \infty$. Since Y_2 is compact, there is a subsequence $\{w_{n'}\}$ such that $i_2(w_{n'})$ converges to a point $y \in Y_2$ as $n' \to \infty$. Clearly (v, y) is in $\overline{i_1 \times i_2(W)}$, hence v is in $\mathrm{Pr}_1(\overline{i_1 \times i_2(W)})$. ∎

LEMMA 2.3 ([9, §1]): *If the Brauer-Manin obstruction to weak approximation is the only obstruction, then the topological closure of $V(K)$ in $\prod_{v \in M_K^\infty} V(K_v)$ is open.*

Proof: For each archimedean absolute value v, the pairing $V(K_v) \times \mathbf{Br}(V)$ is between the connected components of $V(K_v)$ and $\mathbf{Br}(V)$. Hence the image of the projection from $V(\mathbf{A}_K)^{\mathbf{Br}}$ to $\prod_{v \in M_K^\infty} V(K_v)$ consists of connected components, therefore open. Since $\overline{V(K)} = V(\mathbf{A}_K)^{\mathbf{Br}}$, the conclusion follows from Lemma 2.2. ∎

2.2 TATE DUALITY. For any field F, let $\mathrm{Gal}(\bar{F}/F)$ be the Galois group of the algebraic closure $\bar{F}$ over F. For any $\mathrm{Gal}(\bar{F}/F)$-module M, denote by $H^i(F, M)$ the set $H^i(\mathrm{Gal}(\bar{F}/F), M)$. Let A be an abelian variety defined over K whose K-rational points are Zariski dense in A. Let A^t be its dual abelian variety. There exist perfect pairings [16], called Tate pairings,

$$H^0(K_v, A) \times H^1(K_v, A^t) \xrightarrow{<\,,\,>_v} \mathbf{Br}(K_v) \simeq \mathbf{Q}/\mathbf{Z}, \quad \text{for } v \in M_K^f;$$
$$H^0(K_v, A) \times H^1(K_v, A^t) \xrightarrow{<\,,\,>_v} \mathbf{Br}(\mathbf{K_v}), \quad \text{for } v \in M_K^\infty,$$

where $H^0(K_v, A) = A(K_v)$ for $v \in M_K^f$, $H^0(K_v, A) = A(K_v)/A(K_v)^0$ for $v \in M_K^\infty$, $A(K_v)^0$ is the connected component of $A(K_v)$ containing the identity element. Notice that $\mathbf{Br}(\mathbf{R})$ is isomorphic to $\mathbf{Z}/2\mathbf{Z}$ and $\mathbf{Br}(\mathbf{C})$ is trivial. Then we have the following commutative diagram:

$$\begin{array}{lccccc} (2.2) & A(\mathbf{A}_K) & \times & H^1(K, A^t) & \xrightarrow{<,>} & \mathbf{Q}/\mathbf{Z} \\ & \downarrow \pi & & \| & & \downarrow \\ (2.3) & (\prod_{v \in M_K} H^0(K_v, A)) & \times & H^1(K, A^t) & \longrightarrow & \mathbf{Q}/\mathbf{Z} \end{array}$$

where the map π is the canonical projection and the pairing (2.2) is defined as follows:

$$\langle (x_v), a \rangle = \Sigma_{v \in M_K} \langle \pi(x_v), a \rangle_v.$$

The above pairing is well defined because for all but finitely many v, the image of an element $a \in H^1(K, A^t)$ in $H^1(K_v, A^t)$ is zero [10, p. 91].

3. Proof of Theorem 1.1

For simplicity, we only prove Theorem 1.1 for $K = \mathbf{Q}$. The proof for a general number field is similar. We first find the left kernel $A(\mathbf{A_Q})^{H^1}$ of pairing (2.2), then relate it to the left kernel $A(\mathbf{A_Q})^{\mathbf{Br}}$ of pairing (2.1). Using the fact that the $\mathbf{Q}$-rational points are contained in $A(\mathbf{A_Q})^{\mathbf{Br}}$, we conclude the proof.

3.1. Kernel of Pairing (2.2). For an abelian group M, the profinite completion $\lim_{\infty \leftarrow n} M/nM$ is denoted by $\hat{M}$, and M^* denotes $\mathrm{Hom}_{\mathrm{cts}}(M, \mathbf{Q}/\mathbf{Z})$, the group of continuous characters of finite order of M. The Tate–Shafarevich group $\text{Ш}(\mathbf{Q}, A)$ of A over $\mathbf{Q}$ is the kernel of the map $H^1(\mathbf{Q}, A) \to \oplus_{p \le \infty} H^1(\mathbf{Q}_p, A)$. We denote the closure of $A(\mathbf{Q})$ in $A(\mathbf{A_Q})$ (resp. $\prod_{p<\infty} A(\mathbf{Q}_p) \times A(\mathbf{R})/A(\mathbf{R})^0$, $\prod_{p<\infty} A(\mathbf{Q}_p)$, $A(\mathbf{R})$) by $\overline{A(\mathbf{Q})}$ (resp. $\overline{A(\mathbf{Q})}_{fd}$, $\overline{A(\mathbf{Q})}_f$, $\overline{A(\mathbf{Q})}_\infty$).

Proposition 3.1: *Assume that the Tate–Shafarevich groups of A and A^t are finite, then $\overline{A(\mathbf{Q})}_{fd} = (\prod_{p<\infty} A(\mathbf{Q}_p) \times A(\mathbf{R})/A(\mathbf{R})^0)^{H^1}$, i.e. , $\overline{A(\mathbf{Q})}_{fd}$ is the left kernel of the pairing* (2.3) *in the case $K = \mathbf{Q}$.*

Proof: From Tate duality and the assumption that $\text{Ш}(A)$ is finite, we get the Cassels–Tate exact sequence [10, p. 102], [17]:

$$(1) \quad 0 \longrightarrow \widehat{A(\mathbf{Q})} \longrightarrow \prod_{p \le \infty} H^0(\mathbf{Q}_p, A) \xrightarrow{\phi} H^1(\mathbf{Q}, A^t)^* \longrightarrow (\text{Ш}(\mathbf{Q}, A^t))^* \longrightarrow 0,$$

where $H^0(\mathbf{Q}_p, A) = A(\mathbf{Q}_p)$ unless p is archimedean, in which case it is equal to $A(\mathbf{R})/A(\mathbf{R})^0$, and ϕ is induced from the pairing (2.3). From this exact sequence, we see that $\widehat{A(\mathbf{Q})}$ is the left kernel of pairing (2.3) and $\underline{|||}(\mathbf{Q}, A^t)$ is the right kernel.

By Serre [12] and the fact that $A(\mathbf{R})/A(\mathbf{R})^0$ is finite, we see that $\widehat{A(\mathbf{Q})}$ is the topological closure of $A(\mathbf{Q})$ in $\prod_{p<\infty} A(\mathbf{Q}_p) \times A(\mathbf{R})/A(\mathbf{R})^0$. That is $\widehat{A(\mathbf{Q})} = \overline{A(\mathbf{Q})}_{fd}$. ∎

We will use the diagram involving pairings (2.2) and (2.3) in §2.2 to find the left kernel of the pairing (2.2).

PROPOSITION 3.2: *Let A be an abelian variety such that the closure of $A(\mathbf{Q})$ in $A(\mathbf{R})$ is open. If $\underline{|||}(A)$ is finite, then $\overline{A(\mathbf{Q})} = A(\mathbf{A}_\mathbf{Q})^{H^1}$.*

To prove this proposition, we first prove the following lemma.

LEMMA 3.3 (Goursat's Lemma): *Let G and G' be two abelian groups and let Γ be a subgroup of $G \times G'$ for which the two projections $\mathrm{pr}_1 : \Gamma \to G, \mathrm{pr}_2 : \Gamma \to G'$ are surjective. Suppose that there are no proper subgroups $N \lhd G, N' \lhd G'$ such that $G/N \simeq G'/N'$. Then $\Gamma = G \times G'$.*

Proof: Let 1_G and $1_{G'}$ be the identity element of G and G' respectively. Notice that the subgroup generated by $G \times 1_{G'}$ and Γ, or $1_G \times G'$ and Γ, is the whole group $G \times G'$, and

$$\langle G \times 1_{G'}, \Gamma\rangle/\Gamma \simeq (G \times 1_{G'})/\Gamma \bigcap (G \times 1_{G'}),$$

$$\langle \Gamma, 1_G \times G'\rangle/\Gamma \simeq (1_G \times G')/\Gamma \bigcap (1_G \times G').$$

Hence $G/\mathrm{pr}_1(\Gamma \bigcap (G \times 1_{G'})) \simeq G'/\mathrm{pr}_2(\Gamma \bigcap (1_G \times G'))$. By assumption, we have

$$\mathop{\mathrm{pr}}_1(\Gamma \bigcap (G \times 1_{G'})) = G, \quad \mathop{\mathrm{pr}}_2(\Gamma \bigcap (1_G \times G')) = G'.$$

Therefore $\Gamma = G \times G'$. ∎

Remark: There are more general statements of this lemma which we do not need here [2, p. 124], [11, p. 252].

Proof of Proposition 3.2: Suppose that $A(\mathbf{R})$ consists of $s+1$ connected components, $A(\mathbf{R}) = \bigcup_{i=0}^{s} e_i A(\mathbf{R})^0$. Let $X_i = \prod_{p<\infty} A(\mathbf{Q}_p) \times e_i A(\mathbf{R})^0, 0 \le i \le s$. Then X_i is open and closed in $A(\mathbf{A}_\mathbf{Q})$.

In Lemma 2.2, if we let $W = A(\mathbf{Q})\bigcap X_0, Y_1 = \prod_{p<\infty} A(\mathbf{Q}_p), Y_2 = A(\mathbf{R})$, we get that the projection of the closed subgroup $\overline{A(\mathbf{Q})}\bigcap X_0$ to $\prod_{p<\infty} A(\mathbf{Q}_p)$ is $\mathrm{Pr}_f(\overline{A(\mathbf{Q})}\bigcap X_0) = \overline{\mathrm{Pr}_f(A(\mathbf{Q})\bigcap X_0)}$. Similarly, the projection of $\overline{A(\mathbf{Q})}\bigcap X_0$ to $A(\mathbf{R})$ is $\overline{\mathrm{Pr}_\infty(A(\mathbf{Q})\bigcap X_0)}$. By Lemma 3.3, with $G = \overline{\mathrm{Pr}_f(A(\mathbf{Q})\bigcap X_0)}, G' = \overline{\mathrm{Pr}_\infty(A(\mathbf{Q})\bigcap X_0)}$ and $\Gamma = \overline{A(\mathbf{Q})}\bigcap X_0$, we get that

$$\begin{aligned}
\overline{A(\mathbf{Q})}\bigcap X_0 &= \overline{\mathrm{Pr}_f(A(\mathbf{Q})\bigcap X_0)} \times \overline{\mathrm{Pr}_\infty(A(\mathbf{Q})\bigcap X_0)} \\
&= \mathrm{Pr}_f(\overline{A(\mathbf{Q})\bigcap X_0}) \times (A(\mathbf{R})^0 \bigcap \overline{A(\mathbf{Q})_\infty}) \\
&\overset{(a)}{=} \mathrm{Pr}'_f(\overline{A(\mathbf{Q})_{fd}} \bigcap (\prod_{p<\infty} A(\mathbf{Q}_p) \times e_0)) \times A(\mathbf{R})^0 \\
&\overset{(b)}{=} (\prod_{p<\infty} A(\mathbf{Q}_p) \times A(\mathbf{R})^0)^{H^1},
\end{aligned}$$

where Pr'_f is the projection from $\prod_{p<\infty} A(\mathbf{Q}_p) \times A(\mathbf{R})/A(\mathbf{R})^0$ to $\prod_{p<\infty} A(\mathbf{Q}_p)$. The equation (a) uses the assumption that the topological closure of $A(\mathbf{Q})$ is open in $A(\mathbf{R})$; (b) comes from Proposition 3.1 and the definition of the pairing (2.2).

For any $(x_p, x_\infty) \in (\prod_{p<\infty} A(\mathbf{Q}_p) \times A(\mathbf{R}))^{H^1} = A(\mathbf{A_Q})^{H^1}$, let $(x_p, \bar{x}_\infty)$ be its image in the projection to $\overline{A(\mathbf{Q})}_{fd}$. Let $y \in A(\mathbf{Q})$ be such that $\bar{y}_\infty = \bar{x}_\infty$. Then $x_\infty - y_\infty \in A(\mathbf{R})^0$ and $((x_p - y_p), x_\infty - y_\infty) \in (\prod_{p<\infty} A(\mathbf{Q}_p) \times A(\mathbf{R})^0)^{H^1} = \overline{A(\mathbf{Q})}\bigcap X_0$. Hence $((x_p), x_\infty) = ((x_p - y_p), x_\infty - y_\infty) + ((y_p), y_\infty) \in \overline{A(\mathbf{Q})}$. This proves that $A(\mathbf{A_Q})^{H^1} \subseteq \overline{A(\mathbf{Q})}$. Clearly, $A(\mathbf{Q})$ is contained in $A(\mathbf{A_Q})^{H^1}$, so is $\overline{A(\mathbf{Q})}$. Proposition 3.2 is proved. ∎

3.2. $A(\mathbf{A_Q})^{\mathbf{Br}} \subseteq A(\mathbf{A_Q})^{H^1}$. Let G be the Galois group of $\overline{\mathbf{Q}}/\mathbf{Q}$. Let $\mathrm{Div}(A)$ denote the group of divisors of A over $\mathbf{Q}$ and $\mathrm{Div}(\overline{A})$ the group of divisors of $\overline{A} = A \times_{\mathbf{Q}} \overline{\mathbf{Q}}$. $\mathrm{Pic}(\overline{A})$ is the group of divisor classes of A, and $\mathrm{Pic}^0(\overline{A}) = A^t(\overline{\mathbf{Q}})$ the Picard variety of A over $\overline{\mathbf{Q}}$. $NS(\overline{A})$ is the Néron–Severi group of $A \times_{\mathbf{Q}} \overline{\mathbf{Q}}$.

Lemma 3.4 ([7, Theorem 2, p. 403]): *There is an exact sequence:*

$$0 \longrightarrow (\mathrm{Pic}^0(\overline{A}))^G \longrightarrow (\mathrm{Pic}(\overline{A}))^G \longrightarrow NS(\overline{A})^G \xrightarrow{\delta'} H^1(G, \mathrm{Pic}^0(\overline{A}))$$

$$\xrightarrow{\psi} \mathbf{Br}_1(A)/\mathbf{Br}_0(A) \longrightarrow H^1(G, NS(\overline{A})), \tag{2}$$

where

$$\mathbf{Br}_1(A) = \mathrm{Ker}(\mathbf{Br}(A) \longrightarrow \mathbf{Br}(\overline{A})),$$

$$\mathbf{Br}_0(A) = \mathrm{Im}(\mathbf{Br}(\mathbf{Q}) \longrightarrow \mathbf{Br}(A)).$$

Proof: For an abelian variety A, we have

$$0 \longrightarrow \mathrm{Pic}^0(\overline{A}) \longrightarrow \mathrm{Pic}(\overline{A}) \xrightarrow{\Sigma} NS(\overline{A}) \longrightarrow 0. \tag{3}$$

This gives a long exact sequence:

$$\begin{aligned} 0 \longrightarrow (\mathrm{Pic}^0(\overline{A}))^G \longrightarrow (\mathrm{Pic}(\overline{A}))^G \longrightarrow NS(\overline{A})^G \xrightarrow{\delta'} H^1(G, \mathrm{Pic}^0(\overline{A})) \\ \longrightarrow H^1(G, \mathrm{Pic}(\bar{A})) \longrightarrow H^1(G, NS(\overline{A})) \end{aligned} \tag{4}$$

Let $\pi: A \to \operatorname{Spec} \mathbf{Q}$ be the structure morphism. From the Leray spectral sequence

$$H^p(\operatorname{Spec} \mathbf{Q}, R^q\pi_*\mathbf{G}_m) \Longrightarrow H^{p+q}(A_{\acute{e}t}, \mathbf{G}_m),$$

we obtain an exact sequence

$$\cdots \longrightarrow H^2(G, \overline{\mathbf{Q}}^*) \longrightarrow E_1^2 \xrightarrow{\chi} H^1(G, \mathrm{Pic}(\overline{A})) \longrightarrow H^3(G, \overline{\mathbf{Q}}^*), \tag{5}$$

where

$$\begin{aligned} E_1^2 &= \operatorname{Ker}(E^2 \longrightarrow E_2^{0,2}) \\ &= \operatorname{Ker}(H^2(A_{\acute{e}t}, \mathbf{G}_m) \longrightarrow H^0(G, R^2\pi_*\mathbf{G}_m)) \\ &= \operatorname{Ker}(H^2(A_{\acute{e}t}, \mathbf{G}_m) \longrightarrow H^2(\overline{A}_{\acute{e}t}, \mathbf{G}_m)) \\ &= \operatorname{Ker}(\mathbf{Br}(A) \longrightarrow \mathbf{Br}(\overline{A})) = \mathbf{Br}_1(A), \end{aligned}$$

and [1, Ch7, Thm 14] $H^3(G, \bar{\mathbf{Q}}^*) = 0$. Hence

$$\mathbf{Br}_1(A)/\mathbf{Br}_0(A) \overset{\chi}{\simeq} H^1(G, \mathrm{Pic}(\overline{A})).$$

Replacing $H^1(G, \mathrm{Pic}(\overline{A}))$ by $\mathbf{Br}_1(A)/\mathbf{Br}_0(A)$ in (3.4), we get (3.2). ∎

LEMMA 3.5: *Let ρ: $\mathbf{Br_1}(A) \to \mathbf{Br_1}(A)/\mathbf{Br_0}(A)$ be the canonical projection. If $\rho(a) = \psi(a'), x_p \in A(\mathbf{Q}_p), x_\infty \in A(\mathbf{R})$ and e is the identity element in the group $A(\mathbf{Q})$, then*

$$\mathrm{inv}_p a^{(p)}(x_p) - \mathrm{inv}_p a^{(p)}(e) = \langle x_p, a' \rangle_p$$

and

$$\mathrm{inv}_\infty a^\infty(x_\infty) - \mathrm{inv}_\infty a^\infty(e) = \langle x_\infty, a' \rangle_\infty,$$

where $\langle \cdot, \cdot \rangle$ is the Tate pairing in §2.2.

Proof: The proof is essentially a detailed comparison of the definitions of the two pairings. See Manin [7, Proposition 8 c), p. 407]. ∎

COROLLARY 3.6: *There exist inclusions*

$$A(\mathbf{A_Q})^{H^1} \supseteq A(\mathbf{A_Q})^{\mathbf{Br_1}} \supseteq A(\mathbf{A_Q})^{\mathbf{Br}}.$$

Proof: Suppose that $x = ((x_p), x_\infty) \in A(\mathbf{A_Q})^{\mathbf{Br_1}}$. For any $a' \in H^1(G, \bar{A}^t)$, $\psi(a') \in \mathbf{Br_1}(A)/\mathbf{Br_0}(A)$. Since ρ is surjective, there exists $a \in \mathbf{Br_1}(A)$, such that $\rho(a) = \psi(a')$. By Lemma 3.5, $\langle x, a'\rangle = (x, a) = 0$. Hence $x \in A(\mathbf{A_Q})^{H^1}$. This proves the first inclusion. Since $\mathbf{Br_1}(A) \subseteq \mathbf{Br}(A)$, the second inclusion is obvious. ∎

Remark: Notice that both pairings (2.1) and (2.2) come from class field theory. They are compatible [7, p. 407]. We can give an explicit formula for the map χ in (3.5). First [7, p. 403],

$$\mathbf{Br_1}(A) = \operatorname{Ker}(H^2(\mathbf{Q}, \overline{\mathbf{Q}}(A)^*) \longrightarrow H^2(\mathbf{Q}, \operatorname{Div}(A \times_{\mathbf{Q}} \overline{\mathbf{Q}}))).$$

Let $b \in \mathbf{Br_1}(A)$ and $f = (f_{s,t}) \in Z^2(\operatorname{Gal}(K/\mathbf{Q}), K(A)^*)$ representing b, where K is a finite normal extension of $\mathbf{Q}$, then we have $f = \partial D$ for some $D = (D_s) \in C^1(\operatorname{Gal}(K/\mathbf{Q}), \operatorname{Div}(A \times_{\mathbf{Q}} K)) \subseteq C^1(\operatorname{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}), \operatorname{Div}(\overline{A}))$. Then $\chi(f)$ is the cohomology class $[D]$ in $H^1(\operatorname{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}), \operatorname{Pic}(\overline{A}))$ [3, p. 469], [7, p. 410].

3.3. CONCLUSIONS. From Proposition 3.2 and Corollary 3.6, we know that under the assumption of Theorem 1.1, we have

$$A(\mathbf{A_Q})^{\mathbf{Br}} \subseteq A(\mathbf{A_Q})^{H^1} = \overline{A(\mathbf{Q})}.$$

Obviously, $\overline{A(\mathbf{Q})} \subseteq A(\mathbf{A_Q})^{\mathbf{Br}}$. Hence

$$A(\mathbf{A_Q})^{\mathbf{Br}} = A(\mathbf{A_Q})^{H^1} = \overline{A(\mathbf{Q})}.$$

This finishes the proof of Theorem 1.1.

4. Brauer–Manin obstruction is not the only obstruction over K

One naturally asks whether there is a generalization of Conjecture 1.2 to the number field case.

QUESTION 1: *Let V be a smooth irreducible algebraic variety defined over a number field K. Suppose that $V(K)$ is Zariski dense in V. Is the topological closure of $V(K)$ in $\prod_v V(K_v)$ open, where v runs through all the real absolute values? (or more generally, v runs through all the archimedean absolute values).*

If V satisfies weak approximation, or if not, but the Brauer–Manin obstruction is the only obstruction to weak approximation for V, then the answer to this question is affirmative (cf. Lemma 2.1 and [9, §1]). But in general we do not have an affirmative answer to the above question. We give examples of elliptic curves.

PROPOSITION 4.1: *Let E be an elliptic curve defined over $\mathbf{Q}$ with positive Mordell–Weil rank. Suppose that there exists a real quadratic number field K such that the Mordell–Weil rank of E over K is the same as the Mordell–Weil rank of E over $\mathbf{Q}$. Then the topological closure of $E(K)$ in $\prod_{v\in M_K^\infty} E(K_v)$ is not open. Hence the Brauer–Manin obstruction to weak approximation is not the only obstruction.*

Proof: Since $E(\mathbf{Q})$ diagonally imbedded in $\prod_{v\in M_K^\infty} E(K_v)$ is not dense in the identity component, and $E(\mathbf{Q})$ is a subgroup of finite index in $E(K)$, hence $E(K)$ is not dense in the identity component. ∎

Explicit examples can be found using Cremona's book [5].

Example 1: Let E be the elliptic curve whose Weierstrass model is

$$y^2 = x^3 - 50x - 125.$$

Let K be the the real quadratic field $\mathbf{Q}(\sqrt{10})$. The quadratic twist of E by 10 is

$$E^{(10)}\colon y^2 = x^3 - 8x - 8.$$

Then rank $E(K) =$ rank $E(\mathbf{Q}) = 1$, and rank $E^{(10)}(\mathbf{Q}) = 0$.

We also have examples for imaginary quadratic fields.

Example 2: Let E be given by

$$y^2 = x^3 - 8x + 8.$$

Let K be the imaginary quadratic field $\mathbf{Q}(\sqrt{-1})$. The quadratic twist of E by -1 is

$$E^{(-1)}\colon y^2 = x^3 - 8x - 8.$$

Then rank $E(K) =$ rank $E(\mathbf{Q}) = 1$ and rank $E^{(-1)}(\mathbf{Q}) = 0$.

QUESTION 2: *Let V be a smooth irreducible algebraic variety defined over a number field K. Suppose that $V(K)$ is Zariski dense in V. Let $\mathrm{Res}_{K/Q}$ denote the Weil restriction of scalars. Is $(\mathrm{Res}_{K/Q}V)(\mathbf{Q})$ Zariski dense in $\mathrm{Res}_{K/Q}V$?*

The answer is not always affirmative. Again we use the above examples of elliptic curves. $(\mathrm{Res}_{K/Q}E)(\mathbf{Q})$ is canonically identified with $E(K)$ and $(\mathrm{Res}_{K/Q}E)(\mathbf{R})$ is isomorphic to $\prod_{v\in M_K^\infty} E(K_v)$.

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